Lecture Notes

The Fundamental Group

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Summary

The set of path-homotopy classes of paths in a space X does not form a group under the operation $*$, only a groupoid. But if we pick out a point x_0 of X to serve as a "base point" and restrict ourselves to those paths that begin and end at x_0, the set of these path-homotopy classes does form a group under $*$. It will be called the fundamental group *of X. We study the basic properties of the fundamental group and prove that it is a topological invariant.*

1 Definitions and basic properties

We shall prove several properties of the fundamental group. In particular, we shall show that the group is, up to isomorphism, independent of the choice of base point (provided that X is path connected). We shall also show that the group is a topological invariant of the space X, the fact that is of crucial importance in using it to study homeomorphism problems.

Definition 1. Let X be a space; let x_0 be a point of X. A path in X that begins and ends at x_0 is called a *loop* based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the *fundamental group* of X relative to the *base point* x_0. It is denoted by $\pi_1(X, x_0)$.

The operation $*$, when restricted to this set, satisfies the axioms for a group. Given two loops f and g based at x_0, the composition $f * g$ is always defined and it is a loop based at x_0. Associativity, the existence of an identity element $[e_{x_0}]$, and the existence of an inverse $[\bar{f}]$ for $[f]$ are immediate.

Example 1. Let $\mathbb{R}^n$ denote the standard euclidean n-space. Then the group $\pi_1(\mathbb{R}^n, x_0)$ is the trivial group consisting of the identity alone. For if f is a loop in $\mathbb{R}^n$ based at x_0, the straight line homotopy

$$F(s,t) = t x_0 + (1-t) f(s)$$

is a path homotopy between f and the constant loop e_{x_0}.

Example 2. More generally, if X is any *convex* subset of $\mathbb{R}^n$, then $\pi_1(X, x_0)$ is the trivial (one single element) group. The straight-line homotopy will work once again, for convexity of X means that for any two points x and y of X, the straight-line segment

$$\{\, t x_0 + (1-t) y \mid 0 \le t \le 1 \,\}$$

between them lies in X. In particular, the *unit ball* $\mathbb{B}^n$ in $\mathbb{R}^n$,

$$\mathbb{B}^n = \{\, x \mid x_1^2 + \cdots + x_n^2 \le 1 \,\},$$

has trivial fundamental group.

An immediate question one asks is the extent to which the fundamental group depends on the base point. The answer is given in Corollary 1, which follows.

Definition 2. Let α be a path in X from x_0 to x_1. We define a map

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

by the equation

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha].$$

The map $\hat{\alpha}$ is pictured in Figure 1. It is well-defined because the operation $*$ is well-defined. If f is a loop based at x_0, then $\bar{\alpha} *$

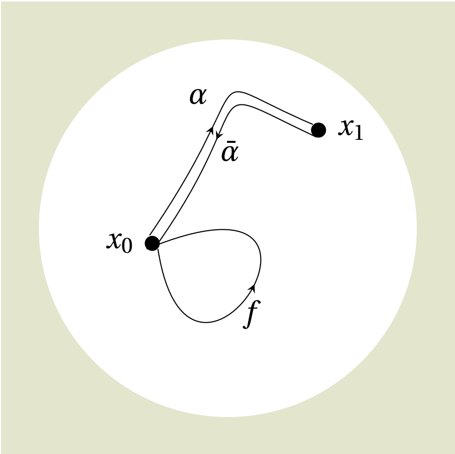

Figure 1

$(f * \alpha)$ is a loop based at x_1. Hence $\hat{\alpha}$ maps $\pi_1(X, x_0)$ into $\pi_1(X, x_1)$, as desired.

Theorem 1. *The map $\hat{\alpha}$ is a group homomorphism.*

Proof. To prove that $\hat{\alpha}$ is a homomorphism, we compute

$$\hat{\alpha}([f]) * \hat{\alpha}([g])$$
$$= ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha])$$
$$= ([\bar{\alpha}] * [f] * [g] * [\alpha]) = \hat{\alpha}([f] * [g]).$$

This proof uses the groupoid properties of $*$. To show that $\hat{\alpha}$ is an isomorphism, we show that if β denotes the path $\bar{\alpha}$, which is the reverse of α, then $\hat{\beta}$ is the inverse for $\hat{\alpha}$. We compute, for each element $[h]$ of $\pi_1(X, x_1)$,

$$\hat{\alpha}(\hat{\beta}([h])) = [\bar{\alpha}] * ([\alpha] * [h] * [\bar{\alpha}]) * [\alpha] = [h]$$

A similar computation shows that $\hat{\beta}(\hat{\alpha}([f])) = [f]$, for each class $[f]$ in $\pi_1(X, x_0)$. ∎

Corollary 1. *If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_0)$.*

Suppose that X is a topological space. Let C be the path component of X containing x_0. It is easy to see that $\pi_1(C, x_0) = \pi_1(X, x_0)$, since all loops and homotopies in X that are based at x_0 must lie in the subspace C. Thus $\pi_1(X, x_0)$ depends only on the path component of X containing x_0, and gives us no information whatever about the rest of X. For this reason, it is usual to deal only with path-connected spaces when studying the fundamental group.

If X is path connected, then all the groups $\pi_1(X, x_)$ are isomorphic, so it is tempting to try to "identify" all these groups with one another, and to speak simply of the fundamental group of the space X, without reference to base point. The difficulty with this approach is that there is no *natural* way of identifying $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$; different paths α and β from x_0 to x_1 may give raise to different isomorphisms between these groups. For this reason, onitting the base point can lead to error.

2 Simply Connected Spaces

Definition 3. A space X is said to be simply connected if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial group for some x_0 in X, and hence for every x in X.

Lemma 1. *In a simply connected space X, any two paths having the same initial and final points are path homotopic.*

Proof. Let f and g be two paths from x_0 to x_1. Then $f * \bar{g}$ is defined and is a loop on X based at x_0. Since X is simply connected, $f * \bar{g} \sim_p e_{x_0}$. Applying the groupoid properties, we see that

$$[(f * \bar{g}) * g] = [e_{x_0} * g] = [g].$$

But

$$[(f * \bar{g}) * g] = [f * (\bar{g} * g)] = [f * e_{x_0}] = [f].$$

Thus f and g are path homotopic. ∎

3 The fundamental group is a topological invariant

It should be intuitively clear that the fundamental group is a topological invariant of the space X. A convenient way to prove this fact formally is to introduce the notion of the "homomorphism induced by a continuous map".

Suppose that $h : X \to Y$ is a continuous map that carries the point x_0 of X to the point y_0 of Y. We often denote this fact by writting

$$h : (X, x_0) \to (Y, y_0).$$

If f is a loop in X based at x_0, then the composite $h \circ f : I \to Y$ is a loop in Y based at y_0. The correspondence $f \mapsto h \circ f$ thus gives rise to a map carrying $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$.

Exercises

1. A subset A of $\mathbb{R}^n$ is said to be *star convex* if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.
   (a) Find a star convex set that is not convex.
   (b) Show that if A is star convex, A is simply connected.
   (c) Show that if A is star convex, any two paths in A having the same initial and final points are path homotopic.
2. Let x_0 and x_1 be two given points of the path-connected space X. Show that the group $\pi_1(X, x_1)$ is abelian if and only if for every par α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.
3. Let $A \subseteq X$ and let $r : X \to A$ be a retraction. Given $a_0 \in A$, show that
   $$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$
   is surjective.
4. Let A be a subset of $\mathbb{R}^n$; let $h : (A, a_0) \to (Y, y_0)$. Show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h_* is the zero homomorphism.

Suggested Readings

- Munkres, James R. *Topology, A First Course.* Prentice Hall, Inc., 1975.
- Willard, Stephen. *General Topology.* Massachusetts: Addison- Wesley, 1970.