The Arzelà-Ascoli Theorem

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3rd March 2014

Abstract

We will form a proof of the Arzelà-Ascoli Theorem through use of the Heine-Borel theorem. We will also be considering some notions of compactness on metric spaces. The Arzelà-Ascoli Theorem then allows us to show compactness, letting us state and prove Peano's existence theorem, pertaining to the existence of the solutions of a type of ODE. Then we will state the Kolmogorov-Riesz compactness theorem, allowing us to show compactness in L^p spaces, building from the Arzelà-Ascoli Theorem.

Contents

1 Introduction

The problem of compactness was first discussed, with relation to limit points, by Bolzano, in his 1817 paper on the intermediate value theorem. At that time, many mathematicians were finding ways of characterising real numbers using sequences, leading the the sequential definition of compactness. Since then, the notion of compactness has been extended to topological spaces. It has been researched by mathematicians such as Weierstass, Lebesgue, Arzelá and Kolmogorov all of whom have made significant advances in the study of compactness (Manya Raman Sundström , 1997).

The Arezelá-Ascoli theorem, relates the notions of compactness and equicontinuity. It it the product of two mathematicians, Arzelá and Ascoli, who were studying both equicontinuity and compactness at a similar time. It was first proven in a weaker form by Ascoli in 1883 then later, in 1893, the proof was completed by Arzelá. Here, we will be showing how the Arzelá-Ascoli theorem can be proven using the Heine-Borel theorem, which we will also prove. Throughout this, we will also be characterising compactness by proving a series of small theorems in metric spaces. This will have the aim of proving that, in metric spaces, the notions of compactness and sequential compactness are equivalent (Manya Raman Sundström (1997), J.W Green and F.A Valentine (1961)).

From the Arzelá-Ascoli theorem, we will then prove the Peano Existence theorem, for the existence of the solutions of ordinary differential equations of the form

$$x'(t) = f(x, x(t)) \tag{1}$$

$$x(t_0) = x_0. \tag{2}$$

This will be followed by the statement of another existance theorem, Cathéodory's Existence theorem, for differential equations of a slightly different form (Gerald Teschl , 2012).

Furthermore, we will be defining L^p spaces, in order to state the Kolmogorov-Riesz compactness theorem, which, as the name suggests, characterises compactness on L^p spaces. Although the proof of this is not stated, it is centered around the use of the Arzelá-Ascoli theorem (Harald Hanche-Olsen and Helge Holden , 2010).

2 Preliminaries

In order to understand many of the following theorems, proofs and definitions, we will require a few basic notions, used commonly in analysis.

2.1 Notation of Sets

The first of these definitions allow us to characterise sets as open and closed in terms of balls which will then lead to a rigorous definition of a limit point. Later on, we will use balls in many situations, allowing us to use the notion of area around a point in n-dimensional space.

Definition. *Open and Closed Balls*

We define the set

$$B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\}, \tag{3}$$

*as the **open ball** of radius r and center x on $\mathbb{R}^n$ and the set*

$$\overline{B_r(x)} = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}, \tag{4}$$

*as the **closed ball** of radius r and center x on $\mathbb{R}^n$. Any open ball with center x, $x \in \mathbb{R}^n$ is called a **neighbourhood** of X*

(George F. Simmons , 1963).

Having defined open and closed balls, we can now give a definitions of open and closed sets respectively.

Definition. *Open Set*
*A set $A \subset \mathbb{R}^n$ is an **open set** if for every point $x \in A$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(x)$ is contained in A*

(George F. Simmons , 1963).

Definition. *Closed Set*
*A set $A \subset \mathbb{R}^n$ is a **closed set** if its compliment, $A^c := \{x \in \mathbb{R}^n : x \notin A\}$ is open*

(George F. Simmons , 1963).

Following form the notions of closed and open sets, we define closure points and limit points.

Definition. *Closure Point*
*A point $A \in \mathbb{R}^n$ is a **closure point** of a set $A \subset \mathbb{R}^n$ if and only if every neighbourhood of x contains al least one point of A. The **closure** of a set A is the set of all closure points of A and is denoted $\overline{A}$*

(George F. Simmons , 1963).

Remark. *A set A is closed if and only if $\overline{A} = A$*

(George F. Simmons , 1963).

Limit points and closure points are very closely related, the difference being that the neighbourhoods of limit points must contain at least one point other that the limit point itself, which can not be said of closure points. From this, it is evident to say that every limit point is a closure point, although the converse is not true.

Definition. *Limit Point (Accumulation point)*
A point x is a ***limit point*** *of a set A if every neighbourhood of x contains at least one point of A that is not x*

(George F. Simmons , 1963).

2.2 Compactness

The notion of compactness recurs commonly in almost every area of analysis and it is one that we will be primarily concerned with throughout this paper. It was first touched upon in 1817 by Bolzano and then returned to in 1877 by Weirstrass, bringing together their respective research to prove the Bolzano-Weirstrass theorem. At the time, Bolzano and Weirstrass had the aim of characterising the properties of real numbers in terms of sequences (Manya Raman Sundström , 1997).

Definition. *Bolzano-Weierstrass Property (Limit Point Compactness)*
Every bounded infinite subset of $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$

(Manya Raman Sundström , 1997).

This is the first and weakest definition of compactness we will use. Later, we will prove that in metric spaces, it is equivalent to sequential compactness, as defined below.

Definition. *Sequential Compactness*
In a Euclidean Space, $\mathbb{R}^n$, A set is ***sequentially compact*** *if and only if every infinite sequence has a convergent subsequence*

(Manya Raman Sundström , 1997).

The notion of sequential compactness is largely characterised by the Bolzano-Weierstass theorem. This is stated without proof, considering it to be known.

Theorem 2.1 (The Bolzano-Weierstrass Theorem)**.** *If a subset $A \in \mathbb{R}^n$ is closed and bounded, it is sequentially compact*

(George F. Simmons , 1963).

In $\mathbb{R}^n$, the notions of compactness and sequential compactness are equivalent, as we will later prove, so here we have a theorem associating compactness with closed bounded sets. Through the relation of closed, boundedness to a type of compactness, it is evident that the Bolzano-Weierstrass theorem will be a powerful tool as we aim to form a proof for the Arzelà-Ascoli theorem.

While Boloanzano and Weiestrass attempted to characterise real numbers in terms of sequences, other mathematicians such as Borel and Lebesgue considered the notion of open covers. Covers proved to be useful, not just in $\mathbb{R}^n$, but in metric spaces and topological spaces also.

Definition. *Cover*
A set, C, is a ***cover*** *of a space A if it is a collection of subsets $C_i \subset A$ whos union is the space A.*

$$A \subseteq \bigcup_{i \in I} C_i \tag{5}$$

(Angus E. Taylor , 1958).

Following this, we may form a new, stronger, definition of compactness which we can apply to all topological spaces.

Definition. *(Countable) Compactness*
We call a space, A, ***compact*** *if and only if each open cover, C_i, for generality, has a finite subcover $K_j \subset C_i$ such that K is finite (where i and j index C and K respectively)*

(Angus E. Taylor , 1958).

2.3 Continuity

The final preliminary notion we must demonstrate is that of equicontinuity. We will begin by evaluating the definition of continuity, before extending it to families of functions. The following is based on Ruth F. Curtain and A.J. Pritchard (1977).

Definition. *Continuity*
A function $f : x \mapsto f(x)$ is ***continuous*** *if and only if*

$$\forall \varepsilon > 0 \ , \exists \delta \ > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \tag{6}$$

We shall denote a **family of functions** $\mathcal{F}$, where $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$. Also, we will denote

$$C([a, b]) := \{f : [a, b] \to \mathbb{R} | \ f \text{ is continuous on } [a, b]\}, \tag{7}$$

as the space of continuous functions.

From this, we can define a type of boundedness unique to families of functions.

Definition. *Uniform Boundedness*
A family of functions, $\mathcal{F} = \{f_1, f_2, \ldots ..f_n | n \in \mathbb{N}\}$, on a set X is ***uniformly bounded*** *if there exists an $m \in \mathbb{N}$ such that f_n is bounded by m, $\forall f_n \in \mathcal{F}$.*

Definition. *Equicontinuity*
A family of functions $\mathcal{F} \subseteq C[a, b]$ is equicontinuous if and only if

$$\forall x_0 \in [a, b], \forall \varepsilon > 0, \ \exists \ \delta = \delta(x, \epsilon) > 0 : |x - x_0| < \delta \tag{8}$$
$$\Rightarrow |f(x) - f(x_0)| < \varepsilon, \forall f \in \mathcal{F}. \tag{9}$$

3 The Heine-Borel Theorem

This section will discuss and prove the Heine-Borel theorem. Although not directly necessary, the Heine-Borel gives a clear route to a proof of the Arzela-Asoli Theorem, as well as being a strong theorem in its own right. We will make use of this in a later section. The proof follows directly from, and can be seen as an extension of, the Bolzano-Weierstrass theorem. In $\mathbb{R}^n$ the Bolzano-Weierstrass theorem is equivalent to one direction of the Heine-Borel theorem and in the next section we will prove this by proving that every sequentially compact metric space is compact.

Theorem 3.1 (The Heine-Borel Theorem)**.** *Take a subset $A \in \mathbb{R}^n$, then A is closed and bounded if and only if it is compact*

(George F. Simmons , 1963).

To prove this, we shall begin by proving that every compact set is closed and bounded. Then, in order to prove that every closed and bounded set is compact, we will reduce the problem to one on a closed bounded box. After assuming that the box is not compact we will use a method of bisection and Cantors intersection theorem (also proved) to form a contradiction.

Lemma 1. *If A is compact, it is bounded.*

Proof. Take the interval $I_n = (-n,\ n)$, so that

$$\mathbb{R} = \bigcup_{n=1}^{\infty} I_n. \tag{10}$$

Then A is covered by $\{I_n\}$ and as A is compact, it will be covered by a finite number of I_n's. The largest of the I_n's is a bound. Hence if A is compact, it is also bounded. □

Lemma 2. *If A is compact, it is closed.*

Proof. Assume the converse, that if A is compact it is not closed. Let p be a limit point of A, not in A. Then for every $q \in A$, let U_q be a neighbourhood of q disjoint from some neighbourhood V_q of p. Then any finite cover $\{U_{q_1}, \ldots, U_{q_n}\}$ is disjoint from $V_{q_1} \cap V_{q_2} \cap \cdots \cap V_{q_n}$, which is a finite union of open sets and hence a neighbouthood of p. Then this is a neighbourhood of p disjoint from A, contradicting that p is a limit point, hence A must be closed. □

Now, we have that if a set is compact, it is closed and bounded. All that remains to prove is that if a set A is closed and bounded, it is compact.

Lemma 3. *A closed subset of a compact set is compact.*

At this point, we look to reduce our proof so that it is only necessary prove that a closed bounded subset of a compact set is compact. Proving the above will allow us to say that a set is compact if a box within that set is compact. We will then be able to use a bisection argument to find an infinite sequence of nesting boxes.

Proof. Let B be a closed subset of a compact set $A \in \mathbb{R}^n$. Let C_B be an open cover of B then $U = \mathbb{R}/B$ is open and $C_B = C_A \cup \{U\}$ is an open cover of A. Since A is compact, then C_A has a finite subcover C'_A which also covers the set B. Since U does not contain any point of B, then B is covered by $C'_B = C'_A \setminus \{U\}$, which is a finite subcollection of the finite collection of C_B. Hence, we can say every open cover C_B has a finite subcover and therefore is compact. □

By this result, we can now reduce the remainder of the proof to showing that a closed, bounded n-dimensional box on $\mathbb{R}^n$ is compact. If the set $A \in \mathbb{R}^n$ is bounded, it can be enclosed in the box

$$B_0 = [-a, a]^n, \tag{11}$$

where $a > 0$. Then, by the above theorem, it is enough to prove that B_0 is compact.

The following proof is difficult to motivate, until we reach the covering lemma. We will be making the assumption that the closed bounded box is not compact, hence an infinite cover has no finite subcover. Then to contradict, we will construct said finite subcover. Firstly though, we use a bisection argument on the box to find an infinite sequence of nesting subboxes, for which we can use Cantor's Intersection theorem.

Lemma 4. *B_0 is compact.*

Proof. Assume, for a contradiction, that B_0 is not compact. Then there exists an infinite open cover C of B_0 that has no finite subcover. If we take bisections of the the box B_0, it can be broken up into 2^n subboxes, each with a diameter half that of B_0. At least one of the subboxes will be covered by an infinite subcover otherwise C would itself have a finite subcover in the union of the finite covers of the subboxes (which we assume it does not). We shall call this infinitely covered subbox B_1. The same logic can be reapplied to yield an infinite sequence of nesting boxes as such:

$$B_0 \supset B_1 \supset \cdots \supset B_k \supset \dots \tag{12}$$

of which the side length of B_k is $2a/2^k$, tending to 0 as $k \to \infty$.

At this point, we have a sequence of nesting boxes in a compact space. We would like to find a point around which we can construct a finite cover of B_k. This will require a lemma, in the form of Cantor's intersection Theorem which we will prove below.

Lemma 5 (Cantor's intersection Theorem)**.** *On a compact space S, a decreasing nesting sequence of non-empty compact subsets of S has a nonempty intersection.*

From the statement of theorem alone, it is easy to see how we will be able to apply it, but first, a proof.

Proof. If we have that

$$C_0 \supset C_1 \supset \cdots \supset C_n \supset \ldots \tag{13}$$

is a decreasing infinite sequence of non-empty compact subsets of S, we can, for a contradiction, suppose that

$$\bigcap C_n = \emptyset. \tag{14}$$

Then let $U_n = X \setminus C_n$. Since $\bigcup U_n = X \setminus \bigcap C_n$ and $\bigcup U_n = \emptyset$, then $\bigcup U_n = X$. As X is compact and U_n is an open cover of it, there exists a finite subcover of U_n. Let U_k be the largest set of this finite cover, then $\bigcap C_n = C_k$, therefore $\bigcup C_n \neq \emptyset$, contradicting the assumption and proving the theorem. □

We can now use Cantor's intersection theorem to say that the intersection of our B_k,

$$B_0 \cap B_1 \cap \cdots \cap B_k \cap \ldots \tag{15}$$

is not empty. So there must exist a point, $p \in B_0$. As C covers B_0, then there exists some $U \in C$ such that $p \in U$. As U is open, there is an n-ball $B(p) \supseteq U$. For a sufficiently large k, we will have that $B_k \supseteq B(p) \supseteq U$ but this means that B_k can be covered by just U, contradicting that C is an infinite cover of B_0, with no finite subcovers. Hence, B_0 must be compact, out required condition □

(Robert Hanson (2004), Paul Vankoughnett (2010) and George F. Simmons (1963)).

To conclude, as we have proven that if a set is compact, it is bounded and closed and that if a set is closed and bounded, it is compact, we have proven the Heine-Borel theorem as written. □

4 Compactness in metric spaces

In the last section, we proved the Heine-Borel Theorem without looking too closely at the notion of compactness. This section will look to give a better insight into compactness, with the ultimate aim of proving that countable compactness and sequential compactness are equivalent in metric spaces.

However, before we begin to prove theorems on metric spaces, we must first define a metric space.

Definition. *Metric*
*Let X be a set, a **metric** (or distance) on X is a function $d : X \times X \to \mathbb{R}$ for which the following conditions are satisfied.*

1. *$d(x, y) \geq 0$ for all $x, y, z \in X$*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$*
3. *$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangle inequality)*

(Ruth F. Curtain and A.J. Pritchard , 1977).

Remark. *The last property is commonly known as the triangle inequality, as it should be obvious that if illustrated graphically, the 3 points x, y and z form a triangle, where the sum of the lengths of any two sides must be greater or equal to the length of the other side.*

Definition. *Metric Space*
*A **metric space** is an ordered pair (X, d) where X is a set and d is a metric on the set*

(Ruth F. Curtain and A.J. Pritchard , 1977).

Remark. *The notation for a a known metric space, (X, d), is often shortened to X alone.*

Now we have a definition for a metric space, we can begin to review notions of compactness.

Definition. *Bolzano-Weierstrass property*
A metric space is said to have the Bolzano Weierstrass property if every infinite sequence has a limit point.

Theorem 4.1. *A metric space is sequentially compact if and only if it has the Bolzano-Weierstrass property.*

Proof. Let X be a metric space. Assume that X is sequentially compact and we look to show that an infinite subset A of X has a limit point. Since A is infinite, we can take a sequence, $\{x_n\} \in A$, of distinct points and this sequence will have a convergent subsequence, as we assumed X is sequentially compact. This sequence will converge to a point x, the limit point of the subsequence and hence a limit point of X.

Now, assume that every infinite subset of X has a limit point. Let $\{x_n\}$ be an arbitrary sequence in X. If $\{x_n\}$ has a point that is infinitely repeated, then it has a constant subsequence which is obviously convergent.

If there is no such point, the set of points in the sequence is infinite and by our assumption, there exists a limit point of this set. Hence the subsequence is convergent and it is sequentially compact □

(George F. Simmons , 1963).

Now we will look toward proving the equivalence of compactness and sequential compactness in metric spaces. We will begin by proving that every compact metric space is sequentially compact. In order to prove that sequentially compact spaces are compact, we will require a few additional definitions.

Theorem 4.2. *Every compact metric space is sequentially compact.*

Proof. As we already have that a metric space is sequentially compact if it has the Bolzano-Weierstrass property, we can use these as equivalent and prove this without much difficulty.

Let X be a compact metric space and A an infinite subset of X. Assume that A has no limit points for a contradiction. Then for every point, $p \in X$, we can form a ball of radius $\varepsilon > 0$, which contains no point other than p. As $A \subset X$, the set of all of these balls will form a cover of A. As X is compact, this cover will have a finite subcover, hence, A must be finite but, A is infinite, so we have a contradiction □

(George F. Simmons , 1963).

In order to prove the other direction, that a sequentially compact metric space is compact, we first must prove a lemma, and with it, introduce the notion of Lebesgue Numbers. As we will see, Lebesgue numbers are a natural property of compact metric spaces and later we will use them to finish our proof that sequential compactness and compactness are equivalent in metric spaces.

Lebesgue Number Lemma. *On a sequentially compact metric space, M, given an open cover $\{C_i\}$, there exists a real number $a > 0$, such that every subset of M, with diameter less than a, is contained within a member of the cover $\{C_i\}$. This a is known as the **Lebesgue number** of the cover.*

Proof. Suppose, for a contradiction, that $\{C_i\}$ is an open cover of M, for which no Lebesgue number exists. Then for any $n \in \mathbb{N}$, there exists some $x_n \in M$, such that $B_{1/n}(x_n) \supseteq c$ is not true for each $C_i \in \{C_i\}$, so that $1/n$ is not a Lebesgue number for $\{C_i\}$. As M is sequentially compact, $\{x_n\}$ has a convergent subsequence, we can call $\{x_{n(r)}\}$, converging to some $x \in M$. As C_i covers M, we have $x \in C_0$ for some $C_0 \in \{C_i\}$ and since C_0 is open, there exists $m \in \mathbb{N}$, such that $B_{2/m}(x) \subseteq U_0$. Now, $B_{1/n}(x)$ contains $x_{n(r)}$ for, say, all $r \geq R$, choose r such that $n(r) \geq m$ and write $s = n(r)$. Then we have that $B_{1/s}(x_s) \subseteq B_{2/m}(x)$ as

$$\begin{aligned} d(x_s, y) &< 1/s && (16) \\ \Rightarrow d(x, y) &\leq d(x, x_s) + d(x_s, y) && (17) \\ &< 1/m + 1/s && (18) \\ &\leq 2/m. && (19) \end{aligned}$$

Hence, we have that $B_{1/s}(x_s) \subseteq U_0$ which we assumed was not true □

(George F. Simmons , 1963).

In the final part of the proof we are working towards, we will be looking to construct an ε-net for a given sequentially compact metric space. Firstly though, we will need to define an ε-net.

Definition. *ε-net*
*On a metric space X, given an $\varepsilon > 0$, a subset A of X is called an ε-**net** if A is finite and*

$$X = \bigcup_{a \in A} B_\varepsilon(a). \tag{20}$$

(George F. Simmons , 1963). From this, we can now define total boundedness.

Definition. *Total boundedness*
*A metric space X is said to be **totally bounded** if it has an ε-net for each $\varepsilon > 0$*

(George F. Simmons , 1963).

Total boundedness, sometimes referred to as precompactness (not to be confused with relative compactness), is a strong notion which we will later use in the Kolmogorov-Riesz theorem.

As we have a sequentially compact set that we wish to find an ε-net for, the natural progression would be to show that all sequentially compact sets are totally bounded, and hence we can form an ε-net.

Theorem 4.3. *Every sequentially compact metric space is totally bounded.*

Proof. Let X be a sequentially compact metric space and $\varepsilon > 0$ given. Take a point $x_1 \in X$ and the corresponding ball, $B_\varepsilon(x_1)$. If this ball contains every point of X, then the set $\{x_1\}$ is an ε-net. For a contradiction, consider, if there exists a point in x, not in $B_\varepsilon(x_1)$. Let that point be x_2 and form the set $B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$. If this contains every point of X, then the set $\{x_1 x_2\}$ is an ε-net, otherwise, continue on to form the union

$$B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \cup \cdots \cup B_\varepsilon(x_n), \tag{21}$$

which will contain every point of X. Then the sequence $\{x_n\}$ would be a sequence with no convergent subsequence. This contradicts the required sequential compactness, so $\{x_1, \ldots, x_n\}$ is an ε-net, hence X is totally bounded □

(George F. Simmons , 1963).

We will use Theorem 4.2, along with the notion of Lebesgue numbers to construct an ε-net in a given sequentially compact metric space where ε, by the definition of a Lebesgue number, is equal to a Lebesgue number of a cover of the set. We can then use this ε-net to show that every cover has a finite subcover.

Theorem 4.4. *Every sequentially compact metric space is compact.*

Proof. Let X be a sequentially compact metric space and let $\{C_i\}$ be an open cover of X. By the Lebesgue Number lemma, we can say that $\{C_i\}$ has a Lebesgue number, a. Also, by Theorem 4.3, there exists $\{x_1, x_2, .., x_n\}$, which is a finite ε-net for X, where $\varepsilon = a$. Let $B_\varepsilon(x_k)$ be the open ball of x_k, the there exists $B_\varepsilon(x_i)$ is contained in some $C_k \in \{C_i\}$. Since

$$X \subseteq \bigcup_{k=1}^{n} B_\varepsilon(x_k) \subseteq \bigcup_{k=1}^{n} C_k, \tag{22}$$

we have a finite subcover of $\{C_i\}$ on X. Hence the set is compact □

(George F. Simmons , 1963).

By this theorem, on all metric spaces, we have proven a one directional version of the Heine-Borel theorem. This can be shown using the Bolzano-Weierstrass Theorem which gives that a closed bounded metric space is sequentially compact. This would then, by Theorem 4.4, imply that a closed bounded metric space is compact.

From what we have proven so far, it is difficult to see how the notions of compactness and sequential compactness could differ. An example of this is the product space $\{0, 1\}^{[0,1]}$, a type of topological space. Using Tychonoff's theorem, this can be proven to be compact. Then, using a method similar to Cantor's diagonalisation argument explained in the next section, it can be found that every subsequence is infinite, therefore it has no convergent subsequneces and is not sequentially compact (Lynn A. Steen and J. Arthur Seebach, Jr , 1970).

5 The Arzelà-Ascoli Theorem

Following a considerable excursion into the nature of compactness, we arrive at the Arzelà-Ascoli Theorem, only to find that we have already proven a large part of it. To begin with, we shall state the theorem.

Theorem 5.1 (The Arzelà-Ascoli Theorem). *A family of functions $\mathcal{F}$ on a metric space, X, is compact if and only if $\mathcal{F}$ is bounded, closed and equicontinuous*

(George F. Simmons , 1963).

The Heine-Borel theorem, states that every $\mathcal{F}$ is closed and bounded if and only if it is compact. It remains it to prove that if $\mathcal{F}$ is compact, it is equicontinuous and that if $\mathcal{F}$ is equicontinuous, it is compact. We begin with the former, as it requires only the definitions of compactness, equicontinuity and an appropriate ε-net.

Proof. First, we assume that $\mathcal{F}$ is compact and let $\varepsilon > 0$ be given.

Recall that $\mathcal{F} \subseteq C[a, b]$ is equicontinuous if and only if

$$\forall x_0 \in [a, b], \forall \varepsilon > 0,\ \exists\ \delta = \delta(x, \epsilon) > 0 : |x - x_0| < \delta \tag{23}$$

$$\Rightarrow |f(x) - f(x_0)| < \varepsilon, \forall f \in \mathcal{F}, \tag{24}$$

which is the form we aim to show the family of functions to be in.

As $\mathcal{F}$ is compact and so is totally bounded, we can find an $(\varepsilon/3)$-net, $\{f_1, f_2, \ldots, f_n\} \in \mathcal{F}$, where each f_i is uniformly continuous. Now define δ to be the to be the minimum of $\{\delta_1, \delta_2, \ldots, \delta_n\}$. If we have $f \in \mathcal{F}$ and choose f_k such that

$$\|f - f_k\| < \varepsilon/3, \tag{25}$$

then

$$d(x, x') < \delta \tag{26}$$

$$\Rightarrow |f(x) - f(x')| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(x')| + |f_k(x') - f(x')| \tag{27}$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \tag{28}$$

This being the condition for equicontinuity, hence this direction is proven □

(George F. Simmons , 1963).

In order to prove that equicontinuity of a closed bounded $\mathcal{F}$ implies its compactness, we will first motivate a method used in the proof, Cantor's diagonalisation argument.

Lemma 6 (Cantor's Diagonalisation Argument)**.** *Consider the set T of all infinite sequences of binary digits, i.e.*

$$T = T_1, T_2, \ldots, T_n \ldots, \tag{29}$$

such that $\forall a \in T_n, a = 0$ or $a = 1$.

If $S_1, S_2, \ldots, S_n, \ldots$ are arbitrary, unique enumerations for T_n, then there will always exist an element $S \in T$ which does not correspond to any S_n in the enumeration. To illustrate this, we construct an enumeration as such, noting the highlighted n^{th} digit,

$$S_1 = \{\mathbf{0}, 0, 0, 0, 0, 0, \ldots\} \tag{30}$$

$$S_2 = \{1, \mathbf{0}, 1, 1, 0, 0, \ldots\} \tag{31}$$

$$S_3 = \{0, 1, \mathbf{0}, 1, 0, 1, \ldots\} \tag{32}$$

$$S_4 = \{1, 1, 0, \mathbf{1}, 1, 0, \ldots\} \tag{33}$$

$$S_5 = \{1, 0, 0, 0, \mathbf{1}, 1, \ldots\} \tag{34}$$

$$S_6 = \{1, 0, 0, 1, 1, \mathbf{0}, \ldots\} \tag{35}$$

$$\ldots \tag{36}$$

For the resulting diagonal sequence, we can take the complementary digit to the n^{th} digit. We call this sequence S and can see that it differs from each S_n as, by construction, their n^{th} digits differ for every n. For example, $S = \{1, 1, 1, 0, 0, 1 \ldots\}$ for the above array.

Remark. *This argument can then be used to show that T is uncountable. If we assume T is countable, then it can be written as an enumeration $S_1, S_2, \ldots, S_n, \ldots$ for all $n \in \mathbb{N}$. As we can construct the sequence $S \neq S_n$ $\forall n \in \mathbb{N}$, such that T is not countable*

(George F. Simmons , 1963).

The last prerequisite for this part of the proof is Caunchy's criterion, which we will be using to show that a sequence we construct is a convergent subsequence. We shall state this, assuming the proof as known.

Definition. *Caunchy's Criterion*
A sequence $\{x_n\}$ converges if and only if for every $\varepsilon > 0$, there exists K such that $|x_n - x_m| < \varepsilon$ when $n, m > K$.

We will now complete the proof of the Arezelá-Ascoli theorem.

Proof. Assume that $\mathcal{F}$ is equicontinuous. We will show that $\mathcal{F}$ is compact by showing every sequence has a convergent subsequence.

By Caunchy's criterion, we can say that it is enough to show that every sequence in $\mathcal{F}$ has a Caunchy subsequence. If we take a subset S_1 of $\mathcal{F}$ such that

$$S_1 = \{f_{1_1}, f_{1_2}, f_{1_3}, \dots\}. \tag{37}$$

As we have assumed that $\mathcal{F}$ is bounded, then there exists an M such that $|f| \leq M \ \forall f \in \mathcal{F}$. Now, if we consider the sequence $\{f_{1_j}(x_2)\}$, we can say that it is bounded, so it has a convergent subsequence by the Bolzano-Weierstrass theorem. Let $S_2 = \{f_{2_1}, f_{2_2}, f_{2_3}, \dots\}$ be a subsequence of S_1 such that $\{f_{2_j}(x_2)\}$. Similarly, we can now consider the sequence $\{f_{2_j}(x_3)\}$, such that $S_3 = \{f_{3_1}, f_{3_2}, f_{3_3}, \dots\}$ is a subsequence of S_2 and this process can continue, in order to construct an array, like in Cantor's diagonalisation argument, as follows:

$$S_1 = \{f_{1_1}, f_{1_2}, f_{1_3}, \dots\} \tag{38}$$
$$S_2 = \{f_{2_1}, f_{2_2}, f_{2_3}, \dots\} \tag{39}$$
$$S_3 = \{f_{3_1}, f_{3_2}, f_{3_3}, \dots\} \tag{40}$$
$$\dots \tag{41}$$
$$S_i = \{f_{i_1}, f_{i_2}, f_{i_3}, \dots\} \tag{42}$$
$$\dots, \tag{43}$$

where each S_i is a subsequence of the one above it. Take the diagonal subsequence of S_i, S, such that $S = \{f_{1_1}, f_{2_2}, f_{3_3}, \dots\}$ and for simplicity, let $S = \{f_1, f_2, f_3, \dots\}$. Then the S differs from every S_i and the sequence $\{f_n(x_i)\}$ is convergent (George F. Simmons , 1963).

As a subsequence such as S can be found like this for any arbitrary sequence in $\mathcal{F}$, we only need to show that S is a Caunchy sequence. Here, we can finally use the remaining assumption of equicontinuity.

Let $\varepsilon > 0$. As $\mathcal{F}$ is equicontinuous, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \varepsilon/3, \tag{44}$$

for all $f_n \in S$.

If we now take a function $f_n(x_i)$ in $\mathcal{F}$ then by our construction, $\exists\ f_m(x_i) \in S$ which differs from $f_m(x_i)$ such that

$$|f_n(x_i) - f_m(x_i)| < \varepsilon/3. \tag{45}$$

Then we can write that

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_m(x) - f_m(x_i)| + |f_m(x_i) - f_n(x_i)| + |f_n(x_i) - f_n(x)| && (46) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. && (47) \end{aligned}$$

Hence the subsequence is Caunchy and so is convergent, meaning that $\mathcal{F}$ is compact, as required □

(D.H.Griffel , 1988).

At the time it was first proven, the significance of the Arzelà-Ascoli theorem was not fully recognised. As compactness was researched more, it became seen as a key notion in analysis and topology, with the Arzelà-Ascoli theorem to characterise it. We will see this in the next two sections.

6 The Caunchy-Peano Existence Theorem

We will now be looking into proving important results that utalise the Arzelà-Ascoli theorem, starting with the Peano-Existence theorem. This theorem proves the existence of the solutions to a common type of initial value problem (IVP). This will, indirectly, allow use of the Arzelà-Ascoli theorem in a much more applied manner. This type of argument will also further motivate the study of compactness, with intent to prove the existence of the solutions of differential equations.

Theorem 6.1 (Caunchy-Peano Theorem)**.** *Let D be an open subset of $\mathbb{R}\times\mathbb{R}$ and $(t_0, x_0) \in D$ and $f : D \to \mathbb{R}$ a continuous function. For an explicit first order differential equation on D,*

$$x'(t) = f(x, x(t)) \tag{48}$$

$$x(t_0) = x_0, \tag{49}$$

there exists a solution, ϕ in a neighbourhood of t_0

(Gerald Teschl , 2012).

Remark. *ϕ is not necessarily unique, an example of this will be given after the proof.*

Peano originally proved this in 1890, before the complete proof of the Arzelà-Ascoli theorem, using a slightly weaker version of the theorem, derived from Ascoli's theorem in 1883. Since Peano's proof, there have been elementary proofs for this theorem such as that in Clifford Gardner (1976). although our proof will be more similar to Peano's proof, utilising the Arzelà-Ascoli theorem (Manya Raman Sundström , 1997).

Proof. We begin be reducing the problem to one on a closed square, as f is continuous on a neighbourhood of (x_0, t_0), there exists $a > 0$ such that f is continuous on the closed square

$$Q = (x, t) \in \mathbb{R} : |x - c| \leq K, |t - c| \leq T, \tag{50}$$

by the fundamental theorem of calculus. We know that ϕ is a solution of the IVP if and only if it satisfies the equation

$$\phi(x) = t_0 + \int_{x_0}^{x} f(t, \phi(t))dt. \tag{51}$$

We define

$$M = \max_{Q} |f(x, t)|, \tag{52}$$

and denote

$$T_1 = \min\{T, K/M\}. \tag{53}$$

Now we look to find a sequence that is uniformly bounded and equicontinuos on this interval, in order to apply the Arzelà-Ascoli theorem. We construct the sequence $\{x_n(t)\}$ on $[0, T_1]$, for each n, define

$$x_n(t) = \begin{cases} c, & \text{for } 0 \leq t \leq T_1/n \\ \\ c + \int_0^{t-T_1/n} f(x_n(s), s)ds, & \text{for } T_1/n < y \leq T_1. \end{cases} \tag{54}$$

This allows us to define $x_n(t)$ recursively, so that later we can form an equicontinuous sequence. Firstly though, we use an induction to show that $x_n(t)$ is uniformly bounded on $[0, T_1]$, by K such that

$$\|x_n - c\| \leq K. \tag{55}$$

We want to show that the above is true for any interval $[0, T_1]$. Take the base case, $[0, T_1/n]$, on which it is trivially true as $x_n(t) = c$. Then we make the inductive assumption that it is true for $[0, k \cdot T_1/n]$ for $(0 \leq k < n)$. Then we must show this holds for $[k \cdot T_1/n, (k+1) \cdot T_1/n]$. By definition,

$$\|x_n - c\| = \left\| \int_0^{t-T_1/n} f(x_n(s), s)ds \right\|. \tag{56}$$

As we have that M is the maximum of $f(x,t)$ and $t - T_1/n$ bounds the integral, we can say

$$\left\| \int_0^{t-T_1/n} f(x_n(s),s)ds \right\| \leq M \cdot |t - T_1/n| \leq M \cdot T_1 \leq K. \tag{57}$$

Hence, $\{x_n(s)\}$ is bounded on $[0,T_1]$.

Now we look to show the equicontinuity of $\{x_n(t)\}$ on $[0,T-1]$. For any $t_1,t_2 \in [0,T_1]$ we can produce the following approximations

$$\|x_n(t_1) - x_n(t_2)\| = \begin{cases} 0, & \text{if } t_1,t_2 \in [0,T_1/n] \\ \| \int_0^{t_2-T_1/n} f(x_n(s),s)ds\|, & \text{if } t_1 \in [0,T_1/n] \text{ and } t_2 \in (T_1/n, T_1], \\ \| \int_0^{t_1-T_1/n} f(x_n(s),s)ds\|, & \text{if } t_2 \in [0,T_1/n] \text{ and } t_1 \in (T_1/n, T_1] \\ \| \int_{t_1-T_1/n}^{t_2-T_1/n} f(x_n(s),s)ds\|, & \text{if } t_1,t_2 \in (T_1/n, T_1]. \end{cases} \tag{58}$$

From the above, we can see that, as it is true in every case,

$$\|x_n(t_1) - x_n(t_2)\| \leq M|t-s|, \tag{59}$$

for any $t_1,t_2 \in [0,T_1]$ which gives us a common constant for the functions in the series $\{x_n(t)\}$, acting as the required δ for the condition of equicontinuity and proving that $x_n(t)$ is equicontinuous.

By the Arzelà-Ascoli theorem, we can now say that $\{x_n(t)\}$ is compact, so there exists a uniformly convergent subsequence, $\{x_{n_i}\}$, that converges to a continuous function $x_\infty(t)$ on $[0,T_1]$ as $n_i \to \infty$. Then we can show that the function $x_\infty(t)$ is the afore mentioned ϕ, a solution to the posed IVP.

For a fixed $t \in (0,T_1]$, we can take n_i to be sufficiently large, such that $T_1/n_i < t$, then, by the definition of $\{x_n(t)\}$, we have

$$x_{n_i}(t) = c + \int_0^t f(x_{n_i}(s),s)ds - \int_0^t f(x_{n_i}(s),s)ds. \tag{60}$$

Finally, as $n_i \to \infty$ and $f(x,t)$ is uniformly continuous, we have

$$\int_0^t f(x_{n_i}(s),s)ds \to \int_0^t f(x_\infty(s),s)ds, \tag{61}$$

and also, as

$$\left| \int_{t-T_1/n_i}^t f(x_{n_i}(s),s)ds \right| \leq \int_{t-T_1/n_i}^t (M)ds, \tag{62}$$

by the definition of M, and

$$M \cdot T_1/n_i \to 0, \tag{63}$$

by the squeeze rule,

$$\left| \int_{t-T_1/n_i}^{t} f(x_{n_i}(s), s) ds \right| \to 0, \tag{64}$$

as $n_i \to \infty$

Evaluating these results, we have that

$$x_\infty(t) = c + \int_0^t f(x_\infty(s), s) ds. \tag{65}$$

This is the solution to the IVP, showing that a solution exists, and therefore proving the theorem □

(Gerald Teschl (2012), Jishan Hu and Wei-Ping Li (2005),Rodrigo Lopéz Pouso (2012)).

In order to see that the solution is not necessarily unique, we shall take an example.

Example 1. *Let $f(x,t) = 25x^{4/5}$. This is continuous and bounded in x, t on $-1 < x < 1$, $t \in \mathbb{R}$. Then by the Peano Existence theorem, the IVP*

$$x' = (x,t), \tag{66}$$
$$x(0) = 0, \tag{67}$$

has at least one solution. Using known techniques, we solve:

$$x'(t) = x^4/5 \tag{68}$$
$$\Rightarrow \int 5x^{-4/5}\ dx = \int 1\ dt \tag{69}$$
$$\Rightarrow 5x^{1/5} = t \tag{70}$$
$$\Rightarrow x(t) = 0\ , \tag{71}$$
$$x(t) = t^5. \tag{72}$$

Hence the IVP has two solutions and the solution is not unique.

The Caunchy-Peano theorem is very similar to another existence theorem, Carathéodory's existence theorem. Cathéodory's theorem applies to the same class of IVPs with f defined on a different domain and can prove the existence of solutions of IVPs with discontinuities. We shall state, but not prove, this theorem below.

Theorem 6.2 (Catheodory's Existance Theorem)**.** *Consider the IVP*

$$x'(t) = f(t, y(t)),\ \textit{with} \tag{73}$$
$$x(t_0) = x_0, \tag{74}$$

on the domain

$$R = \{(x,t) | |x - x_0| \leq a, |t - t_0| \leq b\}. \tag{75}$$

Let $f(t,x)$ be measurable in t for each fixed x and continuous in X for each fixed t. Then there exists a Lebesgue-integrable $m(t)$ such that

$$\|f(t,x)\| \leq m(t) \; \forall (t,x) \in R, \tag{76}$$

then the differential equation has a solution in the neighbourhood of the initial condition

(Ruth F. Curtain and A.J. Pritchard , 1977).

In the next section, we will define measurabillity which will make this theorem somewhat clearer although a greater understanding of measure theory would be required to formulate the proof.

7 Kolmogorov-Riesz Compactness Theorem

In this section we will be stating and explaining the Kolmogorov-Riesz compactness theorem, a powerful theorem in functional analysis, which characterises the notion of total boundedness on L^p spaces. Before we state the theorem, we must first define L^p spaces, which will require some basic measure theory. We define measures on a ring, R, so to begin with, we shall define a ring. These definitions will follow those in Angus E. Taylor (1958).

Definition. *Ring*
A nonempty class of sets R is called a ***ring*** *if for every $E, F \in R$, $E \cup F$ and $E - F$ belong to R*

Definition. *The ring is called a* ***σ-ring*** *if it contains the union of every countable collection of its members.*

Now we can define a measure on the ring.

Definition. *Measure*
On a ring R, a function $\mu : R \to \mathbb{R}$ is a ***measure*** *if and only if*

1. $\mu(E) \geq 0$ *if* $E \in R$
2. $\mu(\emptyset) = 0$
3. *If $\{E_n\}$ is a sequence of pairwise disjoint members of R whose union is in R, then*

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j). \tag{77}$$

Definition. *Measure Space*
A measure space is an ordered pair (X, μ) where X is a set and μ is a measure on the set X.

The L^p spaces, with which the theorem is concerned, are spaces of measurable functions under a the p-norm so we will also require the definitions of a measurable function and a normed space.

Definition. *Measurable function*
A function $f : X \to \mathbb{R}$ is measurable if for every real number a, the set

$$\{x \in X : f(x) > a\} \tag{78}$$

is measurable.

Definition. *Norm*
*A **norm** on a vector space V is a function defined on V with non-negative real number values such that*

1. $\|x\| \geq 0 \ \forall x \in V$
2. $\|x\| = 0 \Leftrightarrow x = 0$
3. $\|ax\| = |a|\|x\| \ \forall a \in \mathbb{R}$
4. $\|x + y\| \leq \|x\| + \|y\| \ \forall x, y \in V$ (the triangle inequality).

Definition. *Normed space*
*A **normed space** is an ordered pair, $(V, \|\cdot\|)$ where V is a vector space and $\|\cdot\|$ is a norm on V.*

Definition. *p - norm*
*For $1 < p < \infty$ and $f \in \mathbb{C}$ measurable on the measure space (X, μ), we define the **p-norm** as*

$$\|f\|_p = \left\{ \int_X |f^p| \ d\mu \right\}^{1/p}. \tag{79}$$

Definition. *L^p space*
Let (X, μ), be a measure space. If $1 \leq p < \infty$, a measurable function f is said to belong to $L^p(\mu)$ if $\|f\|^p$ is integrable.

Examples of L^p spaces include L^0, the space of measurable functions and L^2, the space of square integrable functions. The study of such spaces has numerous applications in physics, engineering and statistics. It is also essential to the study of the existence of solutions to differential equations.

Recall notion of total boundedness stated previously. This allows us to state the theorem.

Theorem 7.1 (Kolmogorov-Riesz Theorem)**.** *Let $1 < p < \infty$. A subset $\mathcal{F}$ of $L^P(\mathbb{R}^n)$ is totally bounded, if and only if*

1. *$\mathcal{F}$ is bounded,*
2. *for every $\varepsilon > 0$, there exists R, such that for every $f \in \mathcal{F}$,*
$$\int_{|x|>R} |f(x)|^p \, dx < \varepsilon^p \tag{80}$$
3. *for every $\varepsilon > 0$ there exists some ρ such that, for every $f \in \mathcal{F}$ and $y \in \mathbb{R}^n$ where $|y| < \rho$,*
$$\int_{\mathbb{R}}^{n} |f(x+y) - f(x)|^p \, dx < \varepsilon^p \tag{81}$$

(Harald Hanche-Olsen and Helge Holden , 2010).

We have now reviewed a number of notions and theorems on compactness, from those on $\mathbb{R}$ to those in topological spaces. We used the Heine-Borel theorem, along with a number of lemmas, to prove the Arzelá-Ascoli theorem. From there we proved Peano's existence theorem, as well as stating two more theorems deriving for the Arzelá-Ascoli theorem. This concludes our discussion of the Arzelá-Ascoli theorem.

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