On commence par calculer $\left(e^x - 1\right)^2$

$$e^x - 1 \underset{x \to 0}{\sim} x$$
$$\left(e^x - 1\right)^2 \underset{x \to 0}{\sim} x^2$$

On cherche maintenant $\sin(2x)^2$

$$\sin(2x) \underset{x \to 0}{\sim} 2x$$
$$\sin(2x)^2 \underset{x \to 0}{\sim} 4x^2$$

On applique la calcul final :

$$\frac{\left(e^x - 1\right)^2}{\sin(2x)^2} \underset{x \to 0}{\sim} \frac{x^2}{4x^2} \underset{x \to 0}{\sim} \frac{1}{4}$$

Donc $\frac{\left(e^x - 1\right)^2}{\sin(2x)^2} \underset{x \to 0}{\to} \frac{1}{4}$

$$\text{On pose} \quad f(x) = \frac{1}{|x|} \qquad \text{et} \qquad g(x) = \frac{1}{x^4}$$

On à $f(x) \underset{x \to 0}{\to} +\infty$ et $g(x) \underset{x \to 0}{\to} +\infty$.
Mais (uniquement si $x \geq 0$) $\frac{f(x)}{g(x)} = \frac{x^4}{x} = x^3 = 0 \neq 1$

$$\text{On pose} \quad f'(x) = \cos(x) - 1 \qquad \text{et} \qquad g'(x) = \sin(x)$$

On à $f'(x) \underset{x \to 0}{\to} 0$ et $g'(x) \underset{x \to 0}{\to} 0$.
Mais

$$\frac{f'(x)}{g'(x)} = \frac{\cos(x) - 1}{\sin(x)} = 0 \neq 1$$

$$\frac{g(x)}{h(x)} = \frac{\frac{1}{x} + \frac{1}{x^2}}{\frac{1}{x} + \frac{2}{x^2}} = \frac{\frac{x+1}{x^2}}{\frac{x+2}{x^2}} = \frac{x+1}{x+2} \underset{x \to +\infty}{\longrightarrow} 1$$

$$\frac{f(x) + g(x)}{f(x) + h(x)} = \frac{\frac{1}{x^2}}{\frac{2}{x^2}} = \frac{1}{2}$$

Soit $a \in \mathbb{R}$

$$
\begin{aligned}
& \mathrm{e}^{f(x)} \underset{x \to a}{\sim} \mathrm{e}^{g(x)} \\
\Leftrightarrow & \frac{\mathrm{e}^{f(x)}}{\mathrm{e}^{g(x)}} \underset{x \to a}{\longrightarrow} 1 \\
\Leftrightarrow & \mathrm{e}^{f(x)-g(x)} \underset{x \to a}{\longrightarrow} 1 \\
\Leftrightarrow & f(x) - g(x) \underset{x \to a}{\longrightarrow} 0 \\
\Leftrightarrow & \frac{f(x)}{g(x)} \underset{x \to a}{\longrightarrow} 1 \\
\Leftrightarrow & f(x) \underset{x \to a}{\sim} g(x)
\end{aligned}
$$