Assignment 6

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1 Homework list

- Section 3.3: 2, 12, 24
- Section 4.1: 26ac
- Section 4.2: 6ab, 12cdef

2 Solution

2 We give a proof by induction. Let $S(n) = 1 + 5 + 9 + ... + (4n - 3)$, where n is a positive integer. We want to prove that for every n, $S(n) = 2n^2 - n$.

Basis step: $S(1) = 2 \times 1^2 - 1 = 1$, which is same with sum of 1.

Inductive step: Assume $S(k) = 2k^2 - k$. We want to show $S(k+1) = 2(k+1)^2 - k$.

$$S(k+1) = 1 + 5 + 9 + ... + (4k - 3) + (4(k+1) - 3) = S(k) + 4(k+1) - 3$$

$$S(k+1) = 2k^2 - k + 4(k+1) - 3 = 2k^2 + 4k + 2 - 1 - k$$

$$2k^2 + 4k + 2 - 1 - k = 2(k^2 + 2k + 1) - 1 - k = 2(k+1)^2 - (k+1)$$

So, we have shown that if $S(k) = 2k^2 - k$, then $S(k+1) = 2(k+1)^2 - k$. Since the statement is also true for the basis case, $S(n) = 2n^2 - n$ for every positive integer n.

12 We give a proof by induction. Let $b(n) = 1/3 + 1/15 + ... + 1/(4n^2 - 1)$, where n is a positive integer.

(a)$b_1 = 1/3, b_2 = 2/5, b_3 = 3/7, b_4 = 4/9, b_5 = 5/11$

(b)$b_n = n/(2n+1)$

(c)We give a proof by induction. We want to prove that for every n, $b_n = n/(2n+1)$.

Basis step: $b_1 = 1/(2 \times 1 + 1) = 1/3$, which is the same with sum of $1/(4 \times 1^2 - 1) = 1/3$.

Inductive step: Assume $b_k = k/(2k+1)$. We want to show $b_{k+1} = (k+1)/(2(k+1)+1) = (k+1)/(2k+3)$.

$$b_{k+1} = b_k + 1/(4(k+1)^2 - 1) = k/(2k+1) + 1/(4(k+1)^2 - 1)$$

$$= k/(2k+1) + 1/(4k^2 + 8k + 3) = k/(2k+1) + 1/((2k+1)(2k+3))$$

$$= k(2k+3)/((2k+1)(2k+3)) + 1/((2k+1)(2k+3) = (2k^2 + 3k + 1)/((2k+1)(2k+3))$$

$$= (k+1)(2k+1)/((2k+1)(2k+3))$$

$$= (k+1)/(2k+3)$$

We have shown that if $b_k = k/(2k+1)$, then $b_{k+1} = (k+1)/(2(k+1)+1) = (k+1)/(2k+3)$. Since the statement is also true for the basis case, $b_n = n/(2n+1)$ for every positive integer n.

(d) Based on our previous proof, the statement is equivalent as b_∞ converges. Since $\lim_{n \to \infty} b_n = \lim_{n \to \infty} n/(2n+1) = 1/2$. The sum of this series converges to 1/2.

24 We give a proof by induction. Let f_n be the nth Fibonacci, where n is a integer ≥ 0. We want to show that for every n, $f_n < 2^n$.

Basis step: $f_0 = 0$, which is less then $2^0 = 1$. $f_1 = 1$, which is less then $2^1 = 2$.

Inductive step: Assume $f_k < 2^k$ and $f_{k+1} < 2^{k+1}$. We want to show $f_{k+2} < 2^{k+2}$.

$$f_{k+2} = f_k + f_{k+1} < 2^k + 2^{k+1} < 2^{k+1} + 2^{k+1} = 2^{k+2}$$

We have shown that if $f_k < 2^k$ and $f_{k+1} < 2^{k+1}$, then$f_{k+2} < 2^{k+2}$. Since the statement is also true for the two basis cases, $f_n < 2^n$ for all n.

26a We give a proof by induction.

We want to prove for all I, $A \cap (\bigcup_{i \in I} A_i) = \bigcup_{i \in I}(A \cap A_i)$.

Basis Step: Assume there are only two sets in A_i family, A_1 and A_2. From basic set identities, we know that $A \cap (A_1 \cup A_2) = (A \cap A_1) \cup (A \cap A_2)$. Thus, the statement is true for basis case.

Inductive step: Assume for any integer $k > 2$, $A \cap (\bigcup_{i \in k} A_i) = \bigcup_{i \in k}(A \cap A_i)$. We are going to show that $A \cap (\bigcup_{i \in k+1} A_i) = \bigcup_{i \in k+1} (A \cap A_i)$.

$$A \cap (\bigcup_{i \in k+1} A_i) = A \cap (\bigcup_{i \in k} A_i \cup A_{k+1})$$

Since $\bigcup_{i \in k} A_i$ can be treated as a single set,

$$\begin{aligned} A \cap (\bigcup_{i \in k} A_i \cup A_{k+1}) &= (A \cap \bigcup_{i \in k} A_i) \cup (A \cap A_{k+1}) \\ &= \bigcup_{i \in k}(A \cap A_i) \cup (A \cap A_{k+1}) \\ &= \bigcup_{i \in k+1} (A \cap A_i) \end{aligned}$$

26c We give a proof by induction.

We want to prove for all I, $(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} A_i^c$.

Basis Step: Assume there are only two sets in A_i family, A_1 and A_2. From basic set identities, we know that $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$. Thus, the statement is true for basis case.

Inductive step: Assume for any integer $k > 2$, $(\bigcap_{i \in k} A_i)^c = \bigcup_{i \in k} A_i^c$.

We are going to show that $(\bigcap_{i \in k+1} A_i)^c = \bigcup_{i \in k+1} A_i^c$.

$$(\bigcap_{i \in k+1} A_i)^c = (\bigcap_{i \in k} A_i \cap A_{k+1})^c$$

Since $\bigcap_{i \in k} A_i$ can be treated as a single set,

$$(\bigcap_{i \in k} A_i \cap A_{k+1})^c = (\bigcap_{i \in k} A_i)^c \cup A_{k+1}^c$$

$$= (\bigcup_{i \in k} A_i^c) \cup A_{k+1}^c$$

$$= \bigcup_{i \in k+1} A_i^c$$

6 (a) not reflexive, symmetric, not antisymmetric, not transitive

(b) reflexive, symmetric, not antisymmetric, not transitive

12 (c) Assume R and S are symmetric and odered pair (a, b) is in $R \cap S$. Then, (a, b) is in R and in S. Since both relations are symmetric, (b, a) is also in R and in S. So (b, a) is in $R \cap S$, which shows $R \cap S$ is symmetric.

(d) Assume R and S are symmetric and odered pair (a, b) is in $R \cup S$. Then, (a, b) is in R or in S. Since both relations are symmetric, (b, a) is also in R or in S. So (b, a) is in $R \cup S$, which shows $R \cup S$ is symmetric.

(e) Assume R and S are transitive and odered pairs (a, b), (b, c) are in $R \cap S$. Then, (a, b), (b, c) are in R and in S. Since both relations are transitive, (a, c) is also in R and in S. So (a, c) is in $R \cap S$, which shows $R \cap S$ is transitive.

(e) Assume R and S are transitive and odered pairs (a, b), (b, c) are in $R \cup S$. Then, (a, b), (b, c) are in R or in S. Since both relations are transitive, (a, c) is in R or in S. So (a, c) is in $R \cup S$, which shows $R \cup S$ is transitive.