# Assignment 4

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## 1 Homework list

- Section 3.1: 18
- Section 3.2: 6, 8, 13
- Section 4.1: 4e, 8, 11, 19, 22ab, 24

## 2 Solution

18 Assume $a$ is an odd integer.By definition, there exists an integer $k$ such that $a = 2k + 1$.

$$a^2 - 1 = (2k+1)^2 - 1 = (4k^2 + 4k + 1) - 1 = 4k^2 + 4k = 4(k^2 + k)$$

Let $p = k^2 + k$. So $a^2 - 1 = 4p$, which means that 4 is a factor of $a^2 - 1$.

6 Based on our previous proof, we grant that $\sqrt{2}$ is not rational. We would prove all following statements by contradiction.

(a) Suppose $-\sqrt{2}$ is rational. By definition, there exists two integers $m, n$ such that $-\sqrt{2} = \frac{m}{n}$. So $\sqrt{2} = \frac{-m}{n}$. Let $p = -m$. $\sqrt{2} = \frac{p}{n}$, where $p$ and $n$ are two integers. It contradicts our assumption that $\sqrt{2}$ is not rational. Thus, $-\sqrt{2} \notin \mathbb{Q}$.

(b) Suppose $1 + \sqrt{2}$ is rational. By definition, there exists two integers $m, n$ such that $1 + \sqrt{2} = \frac{m}{n}$. So $\sqrt{2} = \frac{m-n}{n}$. Let $p = m - n$. $\sqrt{2} = \frac{p}{n}$, where $p$ and $n$ are two integers. It contradicts our assumption that $\sqrt{2}$ is not rational. Thus, $1 + \sqrt{2} \notin \mathbb{Q}$.

(c) Suppose $3 + \sqrt{2}$ is rational. By definition, there exists two integers $m, n$ such that $3 + \sqrt{2} = \frac{m}{n}$. So $\sqrt{2} = \frac{m-3n}{n}$. Let $p = m - 3n$. $\sqrt{2} = \frac{p}{n}$, where $p$ and $n$ are two integers. It contradicts our assumption that $\sqrt{2}$ is not rational. Thus, $3 + \sqrt{2} \notin \mathbb{Q}$.

(d) Suppose $r + \sqrt{2}$ is rational, where r is also a rational number. By definition, there exists two integers $m, n$ such that $r + \sqrt{2} = \frac{m}{n}$. Similarly, there exists two integers $t, k$ such that $r = \frac{t}{k}$ since $r$ itself is a rational number. Then,

$$\sqrt{2} = \frac{m}{n} - \frac{t}{k}$$

$$\sqrt{2} = \frac{km - nt}{nk}$$

Let $p = km - nt$ and $q = nk$. $\sqrt{2} = \frac{p}{q}$, where $p$ and $q$ are two integers. It contradicts our assumption that $\sqrt{2}$ is not rational. Thus, $r + \sqrt{2} \notin \mathbb{Q}$.

8 We give a proof by contradiction. Assume $\log_2(5)$ is rational. By definition,there exists two integers $m, n$ such that $\log_2(5) = \frac{m}{n}$. Then,

$$2^{\log_2(5)} = 2^{\frac{m}{n}}$$

$$5 = 2^{\frac{m}{n}}$$

$$5^n = 2^m$$

Apparently, $5^n$ is odd and $2^m$ is even. $5^n$ can never equal to $2^m$ for any two integers $m, n$. Thus, $\log_2(5)$ is not rational.

13 We give a proof by contradiction. Assume $rx$ is rational, where r is a rational number but $\neq 0$ and x is not rational. By definition, there exists two integers $m, n$ such that $rx = \frac{m}{n}$. Similarly, there exists two integers $t, k$ such that $r = \frac{t}{k}$, since $r$ itself is a rational number. And since $r \neq 0$, $t$ is not 0. Then,

$$rx = \frac{tx}{k} = \frac{m}{n}$$

$$txn = mk$$

$$x = \frac{mk}{tn}$$

Let $p = mk$ and $q = tn$. $x = \frac{p}{q}$, where $p$ and $q$ are two integers. It contradicts our assumption that x is not a rational number. Thus, if $r$ is rational and $\neq 0$ and $x$ is irrational, then $rx$ is irrational.

4e $(A - C) = (A - B) \cup (B - C)$ is not true.

Assume $A = \{1, 2\}$ $B = \{2, 3\}$ $C = \{1, 3\}$

$(A - C) = \{2\}$ $(B - C) = \{2\}$ $(A - B) = \{1\}$

$(A - B) \cup (B - C) = \{1, 2\} \neq (A - C)$

Salveage: $(A - C) \subseteq (A - B) \cup (B - C)$

Proof: Assume $x \in (A - C)$. Then, $x \in A$ and $x \notin C$. Either $x \in B$ or $x \notin B$. So, either $x \in B$ and $\notin C$, or $x \notin B$ and $x \in A$. So $x \in (B - C)$ or $x \in (A - B)$. So $x \in (A - B) \cup (B - C)$. Thus, $(A - C) \subseteq (A - B) \cup (B - C)$

8 (a) Assume $A \subseteq B$. So for all set $X$ that $X \subseteq A$, $X \subseteq B$. Since every element in $P(A)$ must be an subset of $A$, which means it must be an subset of $B$. And every subset of $B \in P(B)$. So every element in $P(A)$ is also an element in $P(B)$. Thus, $P(A) \subseteq P(B)$.

(b) By definition, for all element $X \in P(A \cap B)$, $X \subseteq (A \cap B)$. So, $X \subseteq A$ and $X \subseteq B$. Since $X \subseteq A$, $X \in P(A)$. Similarly, since $X \subseteq B$, $X \in P(B)$. So $X \in (P(A) \cap P(B))$. Thus $P(A \cap B) \subseteq (P(A) \cap P(B))$. Since all previous steps are reversible. $(P(A) \cap P(B)) \subseteq P(A \cap B)$. Thus $P(A \cap B) = P(A) \cap P(B)$.

(c) $P(A \cup B) = P(A) \cup P(B)$

By definition, for all element $X \in P(A \cup B)$, $X \subseteq (A \cup B)$. So, $X \subseteq A$ or $X \subseteq B$. So $X \in P(A)$ or $X \in P(B)$. So $X \in (P(A) \cup P(B))$. Thus $P(A \cup B) \subseteq (P(A) \cup P(B))$. Since all previous steps are reversible. $(P(A) \cup P(B)) \subseteq P(A \cup B)$. Thus $P(A \cup B) = P(A) \cup P(B)$.

11 On the last page.

19 Proof by contrapositive: Assume $B \neq C$. There exists an element $x$ that ($x$ in $B$ but not in $C$) or ($x$ in $C$ but not in $B$). Because $B$ and $C$ are symmetrical in this statement (swatching $B$ and $C$ will give us same proposition). Just assume $x$ in $B$ but not in $C$. For an element $a \in A$, the ordered pair $(a, x)$ will be in $A \times B$ but not in $A \times C$, since $x \notin C$. So $A \times B \neq A \times C$. Thus, if $A \times B = A \times C$, $B = C$.

22a True.

Assume ordered pair $(p, q) \in A \times (B - C)$. So $p \in A$ and $q \in B$ and $q \notin C$. So $(p, q) \in A \times B$ and $(p, q) \notin A \times C$. So $(p, q) \in (A \times B - A \times C)$. Thus, $A \times (B - C) \subseteq A \times B - A \times C$. Since previous steps are all reversible, $A \times B - A \times C \subseteq A \times (B - C)$. So $A \times (B - C) = A \times B - A \times C$.

22b False.

Counterexample:

Assume $A = \{1\}$

$B = \{2\}$

$U = \{1, 2\}$

$A \times B = \{(1, 2)\}$

$U^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

$(A \times B)^c = \{(1, 1), (2, 1), (2, 2)\}$

$A^c \times B^c = \{(2, 1)\}$

Thus, $(A \times B)^c \neq A^c \times B^c$.

24 (a)$\{[0, 1], [1, 2], [2, 3], [3, 4], [4, 5], [5, 6], [6, 7], [7, 8], [8, 9], [9, 10], [10, 11]\}$

(b) $B_1 = (0, 2)$, $B_{10} = (0.9, 1.1)$, $B_{100} = (0.99, 1.01)$