A fuel-optimal mission to an outer planet with an Eve flyby

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Abstract

The parameters of a fuel-optimal mission from Kerbin via Eve to an outer planet with an arbitrary orbit radius were calculated. The Δv gain of performing the Eve flyby instead a direct Hohmann transfer to the outer planet is on the order of 50 m/s. Eccentricity and inclination of planets' orbits were not considered.

Contents

1 Introduction

An outer planet-bound spacecraft initially shares Kerbin's circular orbit around Kerbol, but must somehow raise its apoapsis to at least the orbit radius of the outer planet.

The straightforward way to accomplish this is to burn into a Hohmann transfer orbit straight from Kerbin to the target. Here, however, we shall consider scheduling an Eve encounter along the way to the outer planet.

Going to Eve means that at the start of our journey we throw away perfectly good Δv on lowering our periapsis. However, there are three reasons why this might pay off in the end. Firstly, by diving into the deep gravity well of Eve we can increase the energy worth of our remaining Δv. Secondly, we can get a gravity assist from Eve, accepting some of Eve's own momentum. Thirdly, a transfer orbit from Eve to an outer planet is slightly lower in energy than a transfer orbit from Kerbin since its major axis is a little shorter.

When going to Jool, we can just barely break even in terms of Δv when we do the Eve flyby instead of just the straight Hohmann transfer from Kerbin. The gain in Δv is rather insignificant considering the fewer launch windows, longer travel time and increased effort and risk. However, with the addition of more outer planets to KSP, we can anticipate there will be greater Δv gains to be had from an Eve flyby.

2 Theory

The path of the spacecraft is strung together out of the following five conic section patches.

- Hyperbolic escape from Kerbin.

- Elliptic transfer from Kerbin to Eve.

- Hyperbolic approach to Eve.

- Hyperbolic escape from Eve.

- Elliptic transfer from Eve to outer planet.

We shall consider each patch in reverse order.

2.1 Eve to outer planet transfer

Of all the transfer orbits from Eve to an outer planet, the one that requires the least speed as a spacecraft escapes Eve is the Hohmann transfer. Therefore, it is in terms of Δv the cheapest transfer orbit to enter, whether the spacecraft is coming from Eve's surface or all the way from Kerbin.[1]

The Hohmann transfer orbit is the elliptic orbit that is tangent to both Eve's orbit and the outer planet's orbit. We designate as P_9 the point where the spacecraft leaves Eve and as P_{10} the point where it encounters the outer planet. The velocity at either point is easily found from the vis viva equation,

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[1]Proving this is left as an excercise for the reader.

$$v_{10} = \sqrt{2\mu_S \left(\frac{1}{r_J} - \frac{1}{r_J + r_E} \right)} \, , \qquad (1)$$

$$v_9 = \sqrt{2\mu_S \left(\frac{1}{r_E} - \frac{1}{r_J + r_E} \right)} \, . \qquad (2)$$
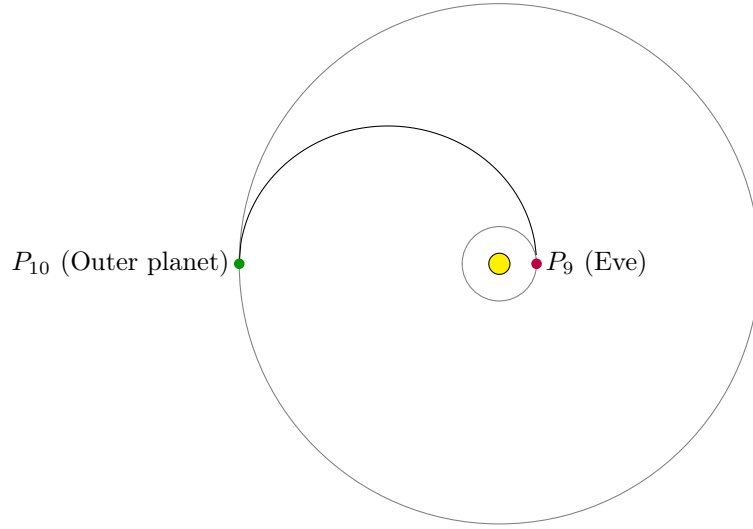

Figure 1: Eve to outer planet transfer orbit.

2.2   Eve escape

The next patch to consider is the hyperbolic escape orbit from Eve. P_8 is where the spacecraft leaves Eve's sphere of influence. P_7 is the periapsis of the orbit. Its distance from Eve's centre is r_{LE}.

To find v_8 from v_9 we translate from Kerbol's frame of reference to Eve's frame of reference, simply adding Eve's orbital velocity v_E,

$$v_8 = v_9 + v_E$$

$$= \sqrt{2\mu_S \left(\frac{1}{r_E} - \frac{1}{r_J + r_E} \right)} + v_E \, . \qquad (3)$$

Next we find v_7 from the conservation of orbital energy,

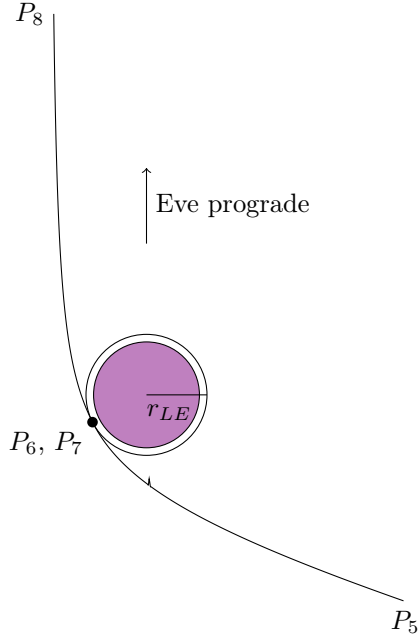$$v_7 = \sqrt{v_8^2 + \frac{2\mu_E}{r_{LE}}} \, . \qquad (4)$$

Figure 2: Eve approach and escape.

2.3 Eve approach

Preceding Eve escape is the Eve approach, a hyperbolic orbit with the same periapsis as the escape orbit. At that periapsis we do a transfer burn, so although P_6 designates the same location as P_7, v_8 and v_7 differ by the velocity added by the burn, v_Δ,

$$v_6 = v_7 - v_\Delta \ . \tag{5}$$

Selecting the optimal value of v_Δ is addressed in subsection 2.6.

P_5 is where the spacecraft enters Eve's sphere of influence. Again, we find v_5 from the conservation of orbital engergy,

$$v_5 = \sqrt{v_6{}^2 - \frac{2\mu_E}{r_{LE}}} \ . \tag{6}$$

Having found the speed v_5 at P_5, let us then find the direction of travel, ϕ_5. We take Eve's prograde direction to be 0 and right turns to be positive when north is up. We have a formula for the angle between the asymptote and the transverse axis of a hyperbolic orbit. Note that since one of the asymptotes points against the direction of travel we must subtract π to find the correct angle.

$$\phi_5 = -\pi + \sec^{-1}\left(\frac{r_{LE}}{\mu_E} \cdot v_5^2 + 1\right) + \sec^{-1}\left(\frac{r_{LE}}{\mu_E} \cdot v_8^2 + 1\right) \ . \tag{7}$$

We can then represent the velocity vector at P_5 in terms of its component along Eve's prograde direction, with magnitude v_{5p}, and its component along Eve's radial out direction, with magnitude v_{5r},

$$v_{5p} = v_5 \cdot \cos \phi_5 \; , \tag{8}$$

$$v_{5r} = v_5 \cdot \sin \phi_5 \; . \tag{9}$$

2.4   Kerbin to Eve transfer

Translating again to Kerbol's frame of reference we find the velocity at P_4, where the spacecraft encounters Eve on the Kerbin–Eve transfer orbit,

$$v_{4p} = v_{5p} + v_E \; , \tag{10}$$

$$v_{4r} = v_{5r} \; . \tag{11}$$

We then find the velocity at P_3, where the spacecraft leaves Kerbin. v_{3p} is found from the conservation of angular momentum and then v_{3r} is found from the conservation of orbital energy,

$$v_{3p} = \frac{r_E}{r_K} \cdot v_{4p} \; , \tag{12}$$

$$\begin{aligned} v_{3r} &= \sqrt{ {v_{4p}}^2 + {v_{4r}}^2 - {v_{3p}}^2 + 2\left({v_K}^2 - {v_E}^2 \right) } \\ &= \sqrt{ \left(1 - \frac{{r_E}^2}{{r_K}^2} \right) {v_{4p}}^2 + {v_{4r}}^2 + 2\left({v_K}^2 - {v_E}^2 \right) } \; . \end{aligned} \tag{13}$$
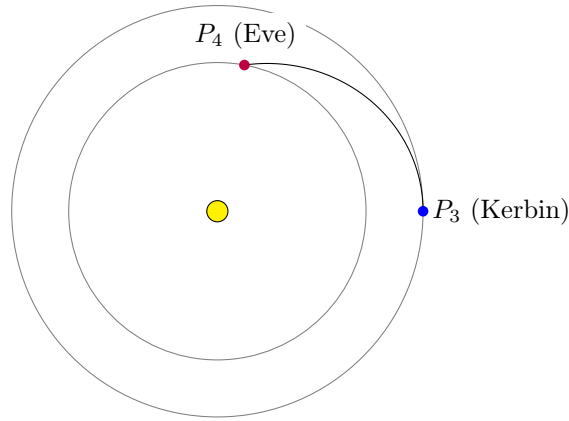

Figure 3: Kerbin to Eve transfer orbit.

2.5 Kerbin escape

Lastly we consider the hyperbolic escape orbit from Kerbin. Its periapsis, designated P_1, is at a distance r_{LK} from Kerbin's centre. P_2 is where the spacecraft escapes Kerbin's sphere of influence. To find the velocity at P_2 we translate from Kerbol's to Kerbin's frame of reference,

$$v_{2p} = v_{3p} - v_K \tag{14}$$

$$v_{2r} = v_{3r} \tag{15}$$

$$v_1 = \sqrt{v_{2p}^2 + v_{2r}^2 + \frac{2\mu_K}{r_{LK}}}$$

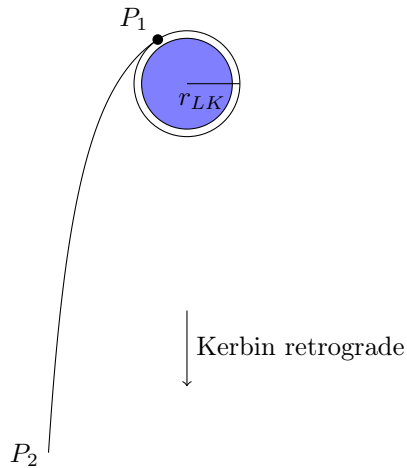$$= \sqrt{(v_{3p} - v_K)^2 + v_{3r}^2 + \frac{2\mu_K}{r_{LK}}} \tag{16}$$

Figure 4: Kerbin escape orbit.

2.6 Optimizing v_Δ

To find the optimal v_Δ we try different values until we find the one that leads to the minimum value of $v_\Delta + v_1$. Equivalently, we can find the value of v_Δ that makes v_{3r} go to zero. The fuel-optimal path has a Kerbin to Eve transfer orbit that is tangent to Kerbin's orbit.[2]

2.7 When to launch

It is preferable when the spacecraft reaches Eve's orbit that Eve is actually situated at that point on its orbit. When we know v_1, we can find the right time to launch for that to happen.

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[2]Proving this is left as another excercise for the reader.

From the vis viva equation we can find the semi-major axis a of the Kerbin to Eve transfer orbit. Then we find its eccentricity e. From these parameters of the transfer orbit's geometry we can find the true anomaly θ and, from that, the eccentric anomaly E of the Eve encounter.[3] The true anomaly of Kerbin at launch is π.

$$a = \frac{2\mu_S r_E}{2\mu_S - r_E v_1{}^2} \; , \tag{17}$$

$$e = \frac{r_K}{a} - 1 \; , \tag{18}$$

$$\theta = -\arccos\left(\frac{a(1 - e^2) - r_E}{e \cdot r_E}\right) \; , \tag{19}$$

$$E = -\arccos\left(\frac{e + \cos\theta}{1 + e\cos\theta}\right) \; . \tag{20}$$

We then find the mean anomaly M from the Kepler equation,

$$M = E - e\sin E \; . \tag{21}$$

We also need to know the orbital periods T_E, T_J, T_{KE} and T_{EJ} of respectively Eve's orbit, the outer planet's orbit, the Kerbin to Eve transfer orbit and the Eve to outer planet transfer orbit.

$$T_{KE} = 2\pi\sqrt{\frac{a^3}{\mu_S}} \; , \tag{22}$$

$$T_{EJ} = 2\pi\sqrt{\frac{(r_J + r_E)^3}{8\mu_S}} \; . \tag{23}$$

$$\tag{24}$$

The travel time from Kerbin to Eve is $(\pi - M) \cdot T_{KE}/2\pi$ and the travel time from Eve to the outer planet is $T_{EJ}/2$. The true anomaly of the Eve encounter is θ and the true anomaly of the outer planet encounter is $\theta + \pi$. Now we can find the true anomalies θ_E and θ_J at which we want respectively Eve and the outer planet at launch time,

$$\theta_E = \theta - \frac{(M - \pi) \cdot T_{KE}}{T_E} \; , \tag{25}$$

$$\theta_J = \theta + \pi - \frac{(M - \pi) \cdot T_{KE} + \pi \cdot T_{EJ}}{T_J} \; . \tag{26}$$

To escape in Kerbin's retrograde direction, we burn when at an angle $-\phi_1$ counterclockwise from the its prograde direction,

$$-\phi_1 = \sec^{-1}\left(\frac{r_{LK}}{\mu_K} \cdot v_1^2 + 1\right) \; . \tag{27}$$

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[3]The relevant equations are given on Wikipedia's page for *True anomaly*.

3 Implementation and results

3.1 Implementation

To calculate the parameters of the described fuel optimal mission a Python script, here included as appendix A, was written. It was run for a range of values of r_J, the radius of the outer planet's orbit.

The mission is assumed to start from Kerbin orbit at an altitude of 100 km. The periapsis of the Eve flyby is also assumed to have an altitude of 100 km.

3.2 Positions of planets at launch

Figure 5 shows how the relevant planets should be positioned at launch. The optimal position of Eve does not vary significantly with r_J over the considered range. For most of the range, launch should be scheduled when Eve is 45.52° behind Kerbin.
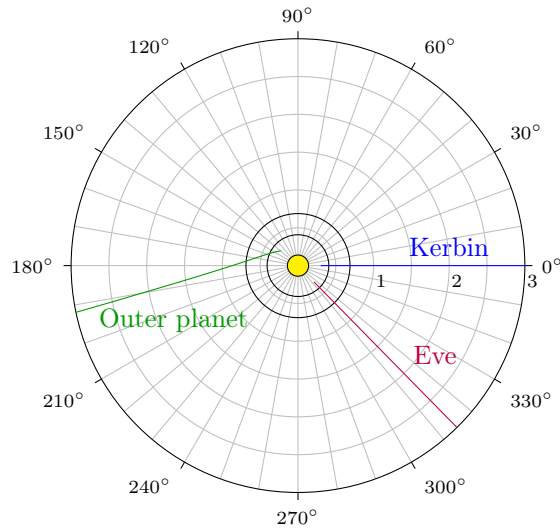

Figure 5: Plots of optimal position at launch as a function of outer planet orbit radius. The scale of the radial dimension is 10^{11} m. Black circles at 0.4 and 0.69 mark orbital radii of Dres and Jool.

3.3 Δv budget

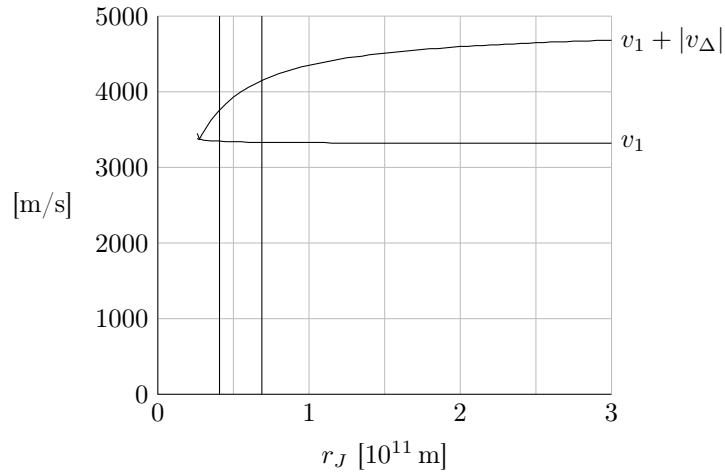Figure 6 shows the optimal v_1 and $v_1 + |v_\Delta|$ as a function of r_J.

Figure 6: Plots of optimal v_1 and $v_1 + |v_\Delta|$ as a function of outer planet orbit radius. Vertical lines at 0.4 and 0.69 mark orbital radii of Dres and Jool.

Figure 7 shows the Δv gain of performing an Eve flyby instead of burning into a direct Hohmann transfer from Kerbin to the outer planet.
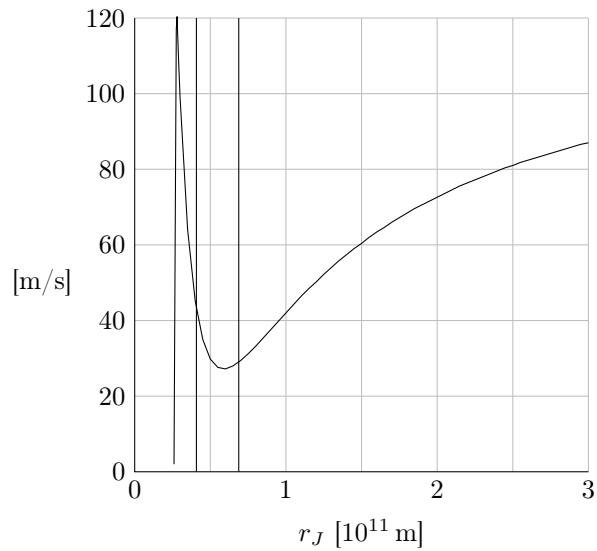

Figure 7: Plot of the gained Δv as a function of outer planet orbit radius. Vertical lines at 0.4 and 0.69 mark orbital radii of Dres and Jool.

A Script for calculations

```python
from math import *
import sys

muS = 1.1723328e18

rLK = 700e3
rK = 13599840256.0
muK = 3.5316e12
vK = sqrt(muS/rK)

rLE = 800e3
rE = 9.83e9
muE = 8.1717302e12
vE = sqrt(muS/rE)
TE = 5657995

rJ = float(sys.argv[1])*1e9
TJ = 2*pi*sqrt(rJ**3/muS)
vJ = sqrt(muS/rJ)
# rJ for Jool is 68.8e9
# TJ for Jool is 104661432

v9 = sqrt(muS*(2/rE - 2/(rE+rJ)))
v8 = v9 - vE
v7 = sqrt(v8**2+2*muE/rLE)

v3r2 = 10.0
vDeltaLo = 0.0
vDeltaHi = v7-sqrt(2*muE/rLE)
vDelta = 0.0
while (v3r2 >= 0.01 or v3r2 < 0):
  vDelta = (vDeltaLo+vDeltaHi)/2
  v6 = v7-vDelta
  v5 = sqrt(v6**2-2*muE/rLE)
  phiEi = -pi+acos(1/(v8**2*rLE/muE+1))+acos(1/(v5**2*rLE/muE+1))
  v5p = v5*cos(phiEi)
  v5r = v5*sin(phiEi)
  v4p = v5p + vE
  v4r = v5r
  v4 = sqrt(v4p**2+v4r**2)
  v3p = v4p*rE/rK
  v3r2 = v4**2-2*muS/rE+2*muS/rK-v3p**2
  if (v3r2 > 0):
    vDeltaLo = vDelta
  else:
    vDeltaHi = vDelta
```

```python
v3r = sqrt(v3r2)
v2p = v3p - vK
v2r = v3r
v2 = sqrt(v2p**2+v2r**2)
v1 = sqrt(v2**2+2*muK/rLK)

a = 2*muS*rE/(2*muS-rE*v1**2)
e = rK/a-1

theta = 2*pi-acos((a*(1-e**2)-rE)/(e*rE))
E = -acos((e+cos(theta))/(1+e*cos(theta)))
M = 2*pi+E-e*sin(E)

TKE = 2*pi*sqrt(a**3/muS)
TEJ = 2*pi*sqrt((rJ+rE)**3/(8*muS))

thetaE = theta-(M-pi)*TKE/TE
thetaJ = theta+pi-((M-pi)*TKE+pi*TEJ)/TJ

phi2 = -acos(v2p/v2)
phi1 = phi2+pi-acos(1/(v2**2*rLK/muK+1))

e1 = 1/cos(phi2-phi1+pi)
a1 = -rLK/(e-1)

vH = sqrt(muS*(2/rK - 2/(rK+rJ)))
v1H = sqrt((vH-vK)**2+2*muK/rLK)

print("vDelta: %.2f m/s"%vDelta)
print("Eve encounter position: %.2f deg"%(theta*180/pi-180))
print("Eve at launch: %.2f deg"%(thetaE*180/pi-180))
print("Outer planet at launch: %.2f deg"%(thetaJ*180/pi-180))
print("phi1: %.2f deg"%(phi1*360/6.28))

print("v1: %.2f m/s"%v1)
print("v1+vDelta: %.2f m/s"%(v1+vDelta))

print("Direct Hohmann transfer: %.2f m/s"%v1H)
print("Gain from Eve flyby: %.2f m/s"%(v1H-v1-vDelta))
```